FACULTY OF SCIENCE

MASTER PROGRAM OF MATHEMATICS

A STUDY ON N-ABSORBING IDEALS OF COMMUTATIVE SEMIRINGS

Prepared By

Ali Tahboub

supervised By

Prof. Mohammad Saleh

Birzeit University

Feb, 2024

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This thesis was defended on ...., 2024. And approved by:

Committee members:

1. Prof. Mohammad Saleh (Head of committee) ....
2. Dr. Khaled Altakhman (Internal Examiner) ....
3. Dr. Khaled Adarbeh (External Examiner) ....

Dedication

This thesis is dedicated to my...

Family, who have always been by my side throughout everything and despite everything.
I would like to extend my dedication to Professor Haynes Miller, who has given me a one of a kind experience in research, and who has been patient and supportive throughout the whole period that I've worked with him.
To Professor Mohammad Saleh, my mentor, who has been very patient despite the fact that I've stopped working on my thesis for a whole year.

Declaration

This thesis, submitted as a requirement for the master's degree of mathematics is my own research except where otherwise acknowledged. This thesis or any part of it hasn't been submitted for a degree in no other institution or university

Ali Tahboub

Signature:

Contents

Abstract

The purpose of this thesis is to investigate n-absorbing ideals of a commutative semiring S, a generalization of the concept of n-absorbing ideals over commutative rings. An ideal I of a commutative semiring S is called n-absorbing ideal of S if and only if whenever $x_1, \cdots, x_{n+1} \in S$ and $x_1 ... x_{n+1} \in I$, then a product of n of these elements is in I.

ملخص

في هذه الرسالة نقوم بدراسة الحلقات نونية الامتصاص في شبه الحلقات التبديلية والتي هي عبارة عن تعميم للمثاليات النونية في الحلقات التبديلية حيث نقوم بدراسة الخواص الاساسية لهذه المثاليات. لتكن ح شبه تبديلية ولتكن م مثالية في ح نقول ان م مثالية نونية الامتصاص اذا كان حاصل ضرب ن+١ من عناصر ح في م فان حاصل ضرب ن من هذه العناصر ايضا في م.

1 Introduction

Semirings are an algebraic structure that are thought of as generalization to rings. Semirings have found their way to some real life applications, mostly in computer science, though not restricted to that.

Throughout this thesis we assume all semirings are commutative semirings with unity $1 \neq 0$. the first formal definition of a semiring is found [1]. This structure became later known as a semiring.

In 1958, Henriksen [2] introduced k-ideals (subtractive ideals) in her PhD thesis. This type of ideals comes to our aid since we do not have the luxury of the cancellation law for addition as we will see in this paper.

Badawi first introduced the concept of 2-absorbing ideals in 2007 [3]. Since then it has been investigated in more details and in further algebraic structures.

In 2012, Darani [4] took the concept of 2-absorbing ideals and worked on it on commutative semirings. In 2012 Ghaudari [5] studied the 2-absorbing ideals in commutative semirings and introduced some of its properties in the quotient semiring and polynomial semiring.

In 2011 Badawi and Anderson published a paper on n-absorbing ideals in commutative rings, which are a generalization to the concept of 2-absorbing ideals, and studied it in details. The idea of n-absorbing ideals itself can be thought of as a generalization of prime ideals. Prime ideals become 1-absorbing ideals by definition as we will see.

L. Sawalmeh studied 2-absorbing ideals over commutative semirings in her thesis under the supervision of Mohammed Saleh (Who is also my supervisor) which was published [6]. Also, I. Murra studied Weakly 1-Absorbing Primary

Ideals over Commutative Semirings in [7].

In this thesis we will study n-absorbing ideals over commutative semirings. In chapter 2, we give the definitions and results that will be needed this thesis. In chapter 3, we study n-absorbing ideals over commutative semirings which is a generalization to results studied in [8].

To clarify, all results obtained in this thesis are generalizations of results in [8].

List of symbols

$\mathbb{N}$	The semiring of natural numbers
$\mathbb{Z}$	The ring of integers
S	Commutative semiring with unit (wherever unspecified)
(n)	the principal ideal generated by n
(m,n)	The ideal generated by n and m
$S[X]$	the polynomial semiring over a semiring
$V(S)$	the set of members of S having an additive inverse
$U(S)$	the set of members of s having a multiplicative inverse
$Rad(I)$	the radical of an ideal I
$Nil(I)$	the nil radical of an ideal I
$Z(S)$	the set of all zero divisors of a semiring S

2 Preliminaries

Definition 2.1 (Semigroup) *A semigroup is an ordered pair* $(M, *)$ *where* M *is a set and* $*$ *is an associative binary operator on* M.

Definition 2.2 (Monoid [1]) *A Monoid is a semigroup* $(M, *)$ *with an identity element* $e \in M$ *such that* $e * m = m * e = m \ \forall m \in M$.

Example 2.3 *The set of positive integers* $\mathbb{Z}^+$ *with usual addition is an example of a semigroup but not a monoid. The set of non-negative integers* $\mathbb{N}$ *with usual addition is a monoid.*

Definition 2.4 (Commutative monoid) *A commutative monoid (Also called an abelian monoid) is a monoid* M *in which* $ab = ba$ *for all* $a, b \in M$.

Remark 2.5 *Given a semigroup* M *that is not a monoid we can embed an element* e *into* M *and define its multiplication by* $e * x = x * e = x$ *for all* $x \in M$ *to get a monoid.*

Definition 2.6 (semiring [1]) *A semiring is an ordered tuple* $(S, +, .)$ *where the following holds :*

1. $(S, +)$ *is a commutative monoid with additive identity* 0.
2. $(S, .)$ *is a monoid with multiplicative identity element* 1.
3. $x.(y + z) = x.y + x.z$ *and* $(y + z).x = y.x + z.x \ \forall x, y, z \in S$.
4. $0.x = x.0 = 0 \ \forall x \in S$.

5. $1 \neq 0$.

All of the above definitions are defined as in [1].

The concept of semirings generalizes the concept of rings which is widely studied in the literature. So naturally, any ring is a semiring.

Example 2.7 *$\mathbb{N}$ with the usual addition is a semiring but not a ring.*

Example 2.8 *Another example of a semiring is the tropical semiring ($\mathbb{R} \cup \{\infty\}, \oplus, \odot$), where $x \oplus y = min\{x, y\}$ and*
$x \odot y = x + y$ is the usual addition on $\mathbb{R}$, and $\infty \odot y = y \odot \infty = \infty$, $\infty \oplus y = y \oplus \infty = y, \forall y \in \mathbb{R} \cup \{\infty\}$.

Example 2.9 *The product of two semirings is also a semiring. If $(S, +_S, \cdot_S)$ and $(T, +_T, \cdot_T)$ is a semiring we can define $(S \times T, +, \cdot)$ where $(a, b) + (c, d) = (a +_S c, b +_T d)$ and $(a, b) \cdot (c, d) = (a \cdot_S c, b \cdot_T d)$*

Similarly, one can define the product of n semirings inductively by defining $S_1 \times \cdots \times S_n = (S_1 \times \cdots \times S_{n-1}) \times S_n$

Definition 2.10 (Subsemiring) *Let $(S, +, \cdot)$ be a semiring. A subset U of S is said to be a subsemiring of S if $(U, +, \cdot)$ is a semiring.*

Proposition 2.11 *Let U be a subset of S. Then U is a subsemiring of S if and only if the following hold :*

1. $0, 1 \in U$.

2. *$a + b \in U$ for all $a, b \in U$.*

3. *$a.b \in U$ for all $a, b \in U$.*

Example 2.12 *Let S be a semiring. Then $P(S) = \{s+1|s \in S\} \cup \{0\}$ is a subsemiring of Ss. First, $P(S)$ is a subset of S. The identities $0 \in P(S)$ and $1 = 1+0 \in P(S)$. $P(S)$ is closed under addition. Let $a, b \in P(S)$. If $a = b = 0$, then $a+b = 0 \in P(S)$. If $a = 0$ and $b \neq 0$, then there exists $s_1 \in S$ such that $b = s_1+1$ and thus $a+b = s1+1 \in P(S)$. Now if $a, b \neq 0$, then there exist s_1 and s_2 such that $a = s_1 + 1$ and $b = s_2 + 1$ and thus $a + b = (s_1 + s_2 + 1) + 1 \in P(S)$. Similarly, $P(S)$ is closed under multiplication. Let $a, b \in P(S)$. if $a = 0$ or $b = 0$ then multiplication is already defined to be 0. other wise, $a = s_1+1$ and $b = s_2+1$. So we have $ab = (s_1+1)(s_2+1) = (s_1s_2+s_1+s_2)+1 \in P(S)$.*

Definition 2.13 (ideal) *A left ideal of a semiring S is a subset I such that the following hold :*

1. *if $a, b \in I$ then $a + b \in I$.*
2. *if $s \in S, x \in I$ then $sx \in I$.*

Definition 2.14 *If S is a semiring and I_1 and I_2 are ideal of the semiring then we define addition and multiplication of the ideals as follows :*

1. *$I_1 + I_2 = \{r + s|r \in I_1, s \in I_2\}$.*
2. *$I_1I_2 = \{r_1s_1 + \cdots r_ns_n|r_i \in I_1, s_i \in I_2, n \in \mathbb{N}\}$.*

Proposition 2.15 *Suppose I_1, I_2 and I_3 are ideals of a semiring S. Then the following holds.[9]*

1. *The sets $I_1 + I_2$ and I_1I_2 are ideals of a semiring S.*
2. *$I_1 + (I_2 + I_3) = (I_1 + I_2) + I_3$ and $I_1(I_2I_3) = (I_1I_2)I_3$.*

3. $I_1 + I_2 = I_2 + I_1$ *and* $I_1 I_2 = I_2 I_1$.

4. $I_1(I_2 + I_3) = I_1 I_2 + I_1 I_3$.

5. $I_1 + I_1 = I_1$, $I_1 + (0) = I_1, I_1 S = I_1, I_1(0) = (0)$ *and* $I_1 + S = S$.

6. *if* $I_1 + I_2 = (0)$ *then* $I_1 = I_2 = (0)$.

7. $I_1 I_2 \subseteq I_1 \cap I_2$ *and if* $I_1 + I_2 = S$, *then* $I_1 I_2 = I_1 \cap I_2$.

8. $(I_1 + I_2)(I_1 \cap I_2) \subseteq I_1 I_2$.

Example 2.16 *let n be a fixed positive number. $n\mathbb{N} = \{ns|s \in \mathbb{N}\}$ is an ideal of the semiring $\mathbb{N}$.*

The above example is a special type of ideals called the principal ideal which is defined as follows:

Definition 2.17 (Principal ideal) *Let S be a semiring, $x \in S$. Then, $xS = \{xs|s \in S\}$ is called a principal ideal and will be denoted by (x).*

An ideal I of a semiring S is called a proper ideal if $I \neq S$.

Definition 2.18 (zero divisor) *An element z of a commutative semiring S is said to be a zero divisor if there is an element $s \in S$ such that $sa = 0$.*

Example 2.19 *In the semiring $\mathbb{N} \times \mathbb{N}$. $(1, 0).(0, 1) = 0$, so both are zero divisors.*

Definition 2.20 (divisors) *Let S be a semiring and let $a, b \in S$. Then we say an element a is a divisor of b if $as = b$ for some element $s \in S$. In that case, we also say a divides b, denoted by $a|b$.*

Definition 2.21 (prime element) *An element p of a semiring S is said to be prime if whenever $p|ab$ for some elements $a, b \in S$ then $p|a$ or $p|b$.*

Definition 2.22 (Prime ideal) *A prime ideal of a semiring S is an ideal I such that whenever H and K are ideals of S and $HK \subseteq I$ then $H \subseteq I$ or $K \subseteq I$.*

Remark 2.23 *In a commutative semiring an ideal is prime iff $xy \in I$ implies that $x \in I$ or $y \in I$.*

Example 2.24 $3\mathbb{N}$ *is a prime ideal in the semiring $\mathbb{N}$.*

Definition 2.25 (Spectrum of a semiring [1]) *Let S be a semiring. The set of all prime ideals in S is called the spectrum of S. and is denoted by $Spec(S)$. So $Spec(S) = \{P \subset S | P$ is a prime ideal of $S\}$.*

We provide an example of an ideal of a semiring that is not a subtractive ideal. But first we define idempotent elements.

Definition 2.26 ([1]) *An element s of a semiring S is said to be additively idempotent if $s + s = s$.*

Definition 2.27 ([1]) *An element s of a semiring S is said to be a multiplicative idempotent if $s.s = s$.*

Example 2.28 *Consider the semiring $S = \{0, 1, u\}$ where u is a multiplicative idempotent and $u + 1 = u = 1 + u$. Let $I = \{0, u\}$, then I is an ideal but not a subtractive ideal since $1 + u = u \in I$ but $1 \notin I$.*

We will also define the multiplicative idempotent ideal as follows:

Definition 2.29 ([1]) *An ideal I is said to be a multiplicative idempotent if $I^2 = I$.*

Simply, called an idempotent ideal.

Example 2.30 *Take the ideal $\mathbb{Z}_{30}$. And take the ideal $I = 6\mathbb{Z}_{30}$. Then I is an idempotent ideal.*

Example 2.31 *Any principal ideal $p\mathbb{N}$ of $\mathbb{N}$ where p is a prime is a subtractive ideal.*

Definition 2.32 (Radical) *The radical of an ideal I of a semiring S is defined as $Rad(I) = \{x \in S | \, x^n \in I \text{ for some positive integer } n\}$.*

Example 2.33 *The radical of the ideal $4\mathbb{N}$ is the ideal $2\mathbb{N}$.*

Definition 2.34 (Maximal ideal) *Suppose S is a semiring. We say that an ideal M is a maximal ideal of S if $M \subseteq I \subseteq S$ for some ideal I of S implies either $M = I$ or $I = S$.*

Example 2.35 *Let $S = \mathbb{Z}_8$ then $2\mathbb{Z}_8$ is a maximal ideal, but $4\mathbb{Z}_8$ isn't.*

Definition 2.36 ([10]) *Let S be a set. An ordering (also called partial ordering) of S is a relation defined on S (written $\leq$) such that:*

1. *$s \leq s$ for all $s \in S$.*
2. *If $s_1 \leq s_2$ and $s_2 \leq s_!$ we have $s_1 = s_2$.*
3. *If $s_1 \leq s_2$ and $s_2 \leq s_3$ we have $s_1 \leq s_3$.*

Definition 2.37 *A set S with and ordering on $\leq$ on S is called an ordered set.*

Definition 2.38 (Totally ordered set [10]) *A partially ordered set S is called a totally ordered set if for any $s_1, s_2 \in S$ we either have $s_1 \leq s_2$ or $s_2 \leq s_1$.*

Definition 2.39 *Let S be an ordered set, and T a subset of S. An upper bound of T (in S) is an element $b \in S$ such that $x \leq b$ for all $x \in T$. An upper bound u of T is called a least upper bound of T if for any other upper bound b of T then $b \leq u$.*

Definition 2.40 *A maximal element $m \in S$ is an element so that for any $x \in S$: if $x \leq s$ then $x = s$.*

Definition 2.41 *A nonempty ordered set S is inductively ordered if every non-empty totally ordered subset has an upper bound.*

Lemma 2.42 (Zorn's lemma [10]) *Let S be a non-empty inductively ordered set. Then there exists a maximal element in S.*

Zorn's lemma is used to show that a chain of ideals has a maximal ideal. But, more importantly, we show the following lemma :

next we will provide some results about prime and maximal ideals over commutative semirings.

Remark 2.43 *Any maximal ideal of a commutative semiring is a prime ideal. However, the converse is not true.*

Example 2.44 *Consider the semiring $S = \mathbb{N}$. Then (0) is a prime ideal since S has no zero divisors. This is an example of a prime ideal that is not maximal.*

Proposition 2.45 *[1] If S is a semiring and I is an ideal of S then the following are equivalent :*

1. I is a prime ideal of S.

2. For any $a, b \in S$ and $ab \in I$ then $a \in I$ or $b \in I$.

3. For any $a, b \in S$ such that $(a)(b) \subseteq I$ then either $a \in I$ or $b \in I$.

Example 2.46 *Consider the semiring $(\mathbb{N}, +, .)$. The ideal $I = \mathbb{N} \backslash \{1\}$ is a prime ideal but the set $I[x]$ of all polynomials with coefficients in I where x is an indeterminate is not prime ideal of $\mathbb{N}[x]$ since $(3 + x)(1 + 2x) = 3 + 7x + 2x^2 \in I[x]$ while neither $3 + x$ nor $1 + 2x$ belong to $I[x]$.*

Theorem 2.47 (Prime Avoidance Theorem for Semirings [11]) *Let S be a semiring, I an ideal, and $P_i (1 \leq i \leq n)$ be subtractive prime ideals of S. If $I \subseteq \bigcup_{i=1}^{n} P_i$ then $I \subseteq P_i$ for some i.*

Corollary 2.48 *Let $P_i, \cdots, P_n$ be incomparable prime ideals of a semiring S. then for each P_i, $P_i \backslash \bigcup_{i=1}^{n} P_i$ is non-empty.*

Definition 2.49 *A nonempty subset W of a semiring S is said to be a multiplicatively closed set (for short an MC-set) if $1 \in W$ and for all $w_1, w_2 \in W$, we have $w_1 w_2 \in W$.*

Lemma 2.50 ([1]) *The maximal elements of the set of all ideals disjoint from an MC-set of a semiring are prime ideals.*

Proof. Suppose S is a semiring and let $M \subseteq S$ be an MC-set. Let Σ be the set of all ideals of S disjoint from M. If $\{I_\alpha\}$ is a chain of ideals in Σ, then $\cup I_\alpha$ is also an ideal disjoint from M and also an upper bound for $\{I_\alpha\}$. Therefore by Zorn's Lemma, Σ has a maximal element. Say P is a maximal element of σ.

We show that P is a prime ideal of S. Take two elements a, b that are not in P. Then we can see that $P \subset P + (a)$ and $P \subset P + (b)$. This implies that $P + (a)$ and $P + (b)$ are ideals of S that are not disjoint from M. So there exist $m_1, m_2 \in M$ such that $m_1 = p_1 + xa$ and $m2 = p_2 + yb$ for some $p_1, p_2 \in P$ and $x, y \in S$. But $m_1 m_2 = p_1 p_2 + p_1 yb + p_2 xa + xyab$. Now it is obvious that if $ab \in P$, then $m_1 m_2 \in P$, which contradicts the fact that P is disjoint from M. so we conclude that $ab \notin P$ and P is indeed a prime ideal of S.

Lemma 2.51 ([1]) *Any semiring S has at least one maximal ideal.*

Lemma 2.52 ([1]) *Any maximal ideal of a semiring S is a prime ideal.*

Definition 2.53 (Primary ideal [1]) *An ideal I of a commutative semiring S is said to be primary iff whenever $xy \in S$ then either $x \in I$ or $y^n \in I$ for some positive number n.*

Remark 2.54 *Any prime ideal is primary.*

Example 2.55 *A primary ideal is not necessarily prime. For example, let $S = \mathbb{N}$, and $I = (9)$. Then I is a primary ideal but not a prime ideal.*

Remark 2.56 *let I be a primary ideal over a semiring S. Then $Rad(I)$ is a prime ideal.*

Proof. Suppose I is a primary ideal of a semiring S. Let $Rad(I)$ be the radical of I. Let $xy \in Rad(I)$ and $x \notin Rad(I)$. Since $(xy)^n = x^n y^n \in I$ for some positive integer n. And since $x^n \notin I$ (other wise $x \in Rad(I)$), then there exists a positive integer m such that $(y^n)^m = y^{mn} \in I$. So $y \in Rad(I)$.

Definition 2.57 (P-primary ideal) *An ideal I of a commutative semiring S is said to be P-primary if $Rad(I) = P$ Where P is a prime ideal of S.*

Definition 2.58 (divided prime ideal) *A prime ideal P of a semiring S is said to be a divided prime ideal if $P \subset xS$ for some $x \in S$.*

Definition 2.59 (Semiring homomorphism) *Let S and T be semirings. Then a semiring homomorphism is a function $f : S \to T$ such that for all $a, b \in S$*

1. $f(a + b) = f(a) + f(b)$.

2. $f(ab) = f(a)f(b)$.

3. $f(0) = 0$.

4. $f(1) = 1$.

Example 2.60 ([1]) *Given any semiring S there is a canonical semiring homomorphism defined as $\alpha : n \to n1_S$.*

Definition 2.61 (ascending chain condition) *A poset is said to satisfy the ascending chain condition if for every weakly increasing sequence $i_1 \leq i_2 \leq \cdots$ there exists a number n such that $i_n = i_{n+1} = \cdots$.*

Definition 2.62 (noetherian semiring) *a semiring S is said to be noetherian iff every chain of ascending ideals satisfies the ascending chain condition.*

Definition 2.63 (semidomain) *A semiring S is called a semidomain if S has no zero divisors.*

3 Results

Definition 3.1 (n-absorbing ideal) *An n-absorbing Ideal I of a semiring S is a proper ideal I such that if $x_1, \cdots, x_{n+1} \in S$ and $x_1 \cdots x_{n+1} \in I$ then the product of n of these elements is in I.*

We can see here that a prime ideal is just a 1-absorbing ideal by definition. So the concept of an n-absorbing ideal could be seen as a generalization to the concept of prime ideals.

Example 3.2 *Consider the semiring of non-negative integers $\mathbb{N}$ with the usual addition and multiplication. Then (2) is n-absorbing for all positive integers n. Let $x_1 \cdots x_{n+1} \in (2)$ then the product is even and so one of the elements is must be even, say x_1, then any n product involving x_1 is going to be in (2).*

The above example should suggest that a prime ideal is n-absorbing for all positive integers n. this is shown in Theorem 3.5(b).

For an Ideal I of a semiring S we might want to address the natural number n such that I is n-absorbing but not m-absorbing for all $m \leq n$ (i.e the least number such that I is n-absorbing). Badawi defines $\omega(I)$ for that exact purpose.

Definition 3.3 *Let S be a semiring and let I be a proper ideal of S. Then $\omega_S(I) = \min\{n : I$ is an n-absorbing Ideal of $S\}$.*

Badawi provides an example of an ideal of a ring that is not n-absorbing for any positive number n.

Working in a semiring can be difficult without additive inverses. However one can make use of k-ideals (Also known as subtractive ideals).

Definition 3.4 (k-Ideals) *An Ideal I of a semiring S is called subtractive (k-ideal) if $a+b \in I$ and $a \in I$ imply $b \in I$.*

In this chapter we generalize the concept of $n-absorbing$ ideals from commutative rings to commutative semirings.

Theorem 3.5 *Let S be semiring. The following holds:*

(a) I is an n-absorbing ideal of S iff for any integer $m > n$, $x_1 \cdots x_m \in I$ implies that there are n of these x_i whose product is also in I.

(b) If I is n-absorbing ideal then I is m-absorbing ideal for all $m > n$.

(c) If I_j is n_j - absorbing ideal for $1 \leq j \leq i$ then $I = I_1 \cap \cdots \cap I_i$ is an n-absorbing ideal for $n = n_1 + \cdots + n_j$.

(d) If $p_1, \cdots, p_n$ are prime elements of a semidomain S then $p_1 \cdots p_n I$ is n-absorbing ideal.

(e) If I is an n-absorbing ideal of S then Rad(I) is n-absorbing ideal .

Proof.

(a) Suppose that for any integer $m > n$, $x_1 \cdots x_m \in I$ implies that there are n of these x_i whose product is also in I. Then Taking $m = n+1$, We get that I is an n-absorbing ideal by definition.To show the converse is also true, We show any n-absorbing is $(n+1)$-absorbing. Let $x_1 \cdots x_{n+2} \in I$ this is iff $(x_1 \cdots x_n)(x_{n+1}x_{n+2})$. Now the product of n of these will be in S, either $(x_{n+1}x_{n+2})$ is among these elements or not. If it is not we are done, if it is, then since S is commutative we have $(\prod_{i \neq r}^{n} x_i)(x_{n+1}x_{n+2}) \in I$. These are $n+1$

terms, since I is n-absorbing n of them have their product in I. as desired. By an induction argument we can show that it is m-absorbing for all $m > n$.

(b) This follows from (a).

(c) We first prove this for the intersection of two Ideals. Let I_1 and I_2 be n-absorbing and m-absorbing ideals. and let $x_1 \cdots x_{n+m+1} \in I_1 \cap I_2$. This implies that n of these elements have their product in I_1. Let N be the set containing exactly these elements. Let M be the set containing the m elements whose product are in I_2. Now define $U = M \cup N$. Note that U will have at most $m+n$. Next note that $\prod_{x \in U} x \in I_1 \cap I_2$. If this product has $m+n$ unique elements we are done. Else we can multiply by any elements not in U until the length of the product is $n+m$. To show this works for the intersection of n ideals, let the statement hold up to the intersection of k ideals. Let $I_1, \cdots, I_{k+1}$ be ideals where I_j is $n_j - absorbing$. Let $I = I_1 \cap \cdots \cap I_{k+1}$. But $I = (I_1 \cap \cdots I_k) \cap I_{k+1}$ and $(I_1 \cap \cdots \cap I_k)$ is $n_1 + \cdots + n_k$-absorbing. and so I must be $n_1 + \cdots + n_{k+1}$ absorbing.

(d) Let $x_1, \cdots, x_{n+1}$ be elements of S such that $x_1 \cdots x_{n+1} \in p_1 \cdots p_n I$. Then $p_1 \ldots p_n s = x_1 \cdots x_{n+1}$ for some $s \in I$. Note that for each $1 \leq i \leq n$ we have $p_i(\prod_{j \neq i} p_j)s = x_1 \cdots x_{n+1}$. This would imply that for each $1 \leq i \leq n$ we have $p_i | x_k$ for some $1 \leq k \leq n+1$. That there must exist a product of at most n elements of $x_1, \cdots, x_{n+1} \in I$.

(e) Let I is n-absorbing. Note that if $x \in Rad(I)$ then $x^n \in I$. Let $x_1 \cdots x_{n+1} \in Rad(I)$. Then we have $x_1^n \cdots x_{n+1}^n = (x_1 \cdots x_{n+1})^n \in I$.and thus we have since I is an n-absorbing ideal we might assume $x_1^n \cdots x_n^n \in I$, in other words we have $x_1 \cdots x_n \in Rad(I)$.

Example 3.6 *The converse of (b) is not true. In the semiring* $(\mathbb{N}, +,)$, *let* $I =< 4, 5 >$. *Then* $I = \{0, 4, 5, 8, 9, 10, 12, 13, 14, \cdots\} = \mathbb{N} \backslash \{1, 2, 3, 6, 7, 11\}$ *is a* 2*-absorbing ideal but not a prime ideal. We show* I *is two absorbing by contradiction. Let* $abc \in I$ *for some* $a, b, c \in \mathbb{N}$ *and* $a, b, c \neq 1$ *(If one of them is 1 then the product of two is in* I*). If no product of two elements is in* I *then* ab, bc *and* $ac \in \{1, 2, 3, 6, 7, 11\}$ *So* $ab = bc = ac = 6$ *which leads to* $c^2 = 6$ *a contradiction.*

Some of the results from the previous theorem can be rephrased with the ω function. For instance (c) becomes : $\omega(I_1 \cap \cdots \cap I_n) \leq \omega(I_1) + \cdots + \omega(I_n)$, In particular, if $P_1 \cdots P_n$ are prime ideals then $\omega(P_1 \cap \cdots \cap P_n) \leq n$. (e) becomes: if $\omega(I) \leq n$ then $\omega(Rad(I)) \leq n$.

now we want to give an example when the equality in (c) is strict, that is, $\omega(I_1 \cap \cdots \cap I_n) = \omega(I_1) + \cdots + \omega(I_n)$. That is we want $\omega(P1 \cap P2) = 2$. So consider $\mathbb{Z}_6$. We have exactly two prime ideals in $\mathbb{Z}_6$ and these are $P_1 = \{0, 3\}$ and $P_2 = \{0, 2, 4\}$. Note that $P_1 \cap P_2 = \{0\}$ which is not prime since $2 * 3 = 0$. But the zero ideal will be 2-absorbing since 2 and 3 are zero divisors.

More generally, equality in part (c) is met when $P_1, \cdots, P_n$ are incomparable prime ideals.

Remark 3.7 *Let* $P_1, \cdots, P_n$ *be incomparable prime ideals then* $\omega(P_1 \cap \cdots \cap P_n) = \omega(P_1) + \cdots + \omega(P_n)$.

To show that, take $x_i \in P_i \backslash \bigcup_{j \neq i} P_j$ i.e we want to take an element from each prime ideal that only belongs to that prime ideal (Such choice is possible by corollary 3.7) . then $x_1 \cdots x_n \in P_1 \cap \cdots \cap P_n$ but no proper sub product of $x_1, \cdots, x_n$ is in $P_1 \cap \cdots \cap P_n$ (otherwise, this would imply $x_i \in P_j$ and $i \neq j$ which contradicts the choice of x_i's). this would imply that $\omega(P_1 \cap \cdots \cap P_n) \geq n$ and so there is a strict equality.

Definition 3.8 (minimal prime ideal over I) *Let S be a semiring and let I be an ideal of S. We say that a prime ideal P is minimal over I if P is minimal among all prime ideals containing I.*

For the next theorem, if S is a semiring we let $Min_S(I)$ Denote the set of prime ideals minimal over S.

Theorem 3.9 *If I is an n-absorbing ideal of a semiring S. Then there are at most n k-prime ideals minimal over S. Moreover $|Min_R(I)| \leq \omega(I)$.*

Proof. If $n = 1$ then the result trivially holds since I itself is a prime. So we may assume $n \geq 2$. Let $P_1, \cdots, P_{n+1}$ be distinct minimal prime k-ideals over I. Then for each $1 \leq i \leq n$ we can choose $x_i \in P_i \backslash ((\bigcup_{k \neq i} P_k) \cup P_{n+1})$. we can choose for each P_i a $c_i \in S - P_i$ such that $c_i x_i^{n_i} \in I$ for some $n_i \geq 1$. Since $I \subseteq P_{n+1}$ is an n-absorbing k-ideal we get that $c_i x_i^{n-1} \in I$ (otherwise we get $x_i^n \in I \subseteq P_{n+1} \implies x_i \in P_{n+1}$) for all $l \leq i \leq n$. and so $(c_1 + \cdots c_i) x_1^{n-1} \cdots x_n{}^{n-1} \in I$. Since $x_i \in P_i - (\bigcup_{k \neq i} P_k)$ and $c_i x_i^{n-1} \in I \subseteq P_1 \cap \cdots \cap P_n$. We have that $c_i \in (\bigcap_{k \neq i} P_i) - P_i$ (To see this note that if $c_i \notin (\bigcap_{k \neq i} P_k) - P_i$ that would imply $x_i \in P_k, i \neq k$ which is a contradiction to the choice of x_i). Now since Each P_i is assumed to be a k-ideal we can conclude that $c_1 + \cdots + c_n \notin P_i, 1 \leq i \leq n$. Otherwise, since $\sum_{k \neq i} c_k \in P_i$ this would imply $c_i \in P_i$.Hence we have $(c_1 + \cdots + c_n) \prod_{k \neq i} x_k \notin P_i$ for $1 \leq i \leq n$ and so $(c_1 + \cdots + c_n) \prod_{k \neq i} x_k \notin I$. The preceding argument shows that $x_1 \cdots x_n \in P_{n+1}$(Since I is n-absorbing). But is should imply that $x_i \in P_{n+1}$ for some i and thus a contradiction. The moreover statement is self evident.

Example 3.10 *As in [8] We want to give examples of an ideal I of a semiring that is not a ring with the following property, I is n-absorbing but not $n-1$-absorbing, and for $m \leq n$ we want exactly m minimal ideals over I. First the ideal $8\mathbb{N}$ is 3-absorbing but not 2-absorbing since $2.2.2 \in 8\mathbb{N}$ but $4 \notin \mathbb{N}$. $8\mathbb{N}$ has exactly one minimal prime ideal over it, that is $2\mathbb{N}$. Another example is $12\mathbb{N}$ which is also 3-absorbing but not 2-absorbing and has exactly 2 minimal prime ideals over it, that is $2\mathbb{N}$ and $3\mathbb{N}$. finally we have $42\mathbb{N}$ which is also 3-absorbing but not 2-absorbing and has $2\mathbb{N}$, $3\mathbb{N}$ and $7\mathbb{N}$ minimal over it.*

Remark 3.11 *The following property of comaximal ideals holds in semirings just like in rings: If $\{I_k\}_{k=1}^n$ is a set of comaximal prime ideals then $\bigcap_{i=1}^{n} I_k = \prod_{i=1}^{n} I_k$. [9]*

Proof. First We show if I_1, I_2 are comaximal then $I_1 \cap I_2 = I_1 I_2$. This is straightforward since $I_1 \cap I_2 = (I_1 \cap I_2)(I_1 + I_J) \subseteq I_1 I_2 \subseteq I_1 \cap I_2$. Let $\{I_k\}_{k=1}^n$ be a set of comaximal Ideals. Define J to be the intersection of the first $n-1$ ideals. that is, define $J = \bigcap_{i=1}^{n-1} I_k$. We will show that J and I_n are comaximal. We show that by contradiction. Suppose J, I_n are not comaximal.Then $J + I_n$ is a proper ideal of S and a result contained in a maximal ideal m of S. this means that J is contained in m which implies that $I_k \subseteq m$ for some $1 \leq k \leq (n-1)$. We now get that $I_k + I_n \subseteq m$, this contradicts the assumption that $\{I_k\}_{k=1}^n$ is a set of comaximal ideals. So J and I_n are comaximal and $J \cap I_n = J \cap I_n$. Using mathematical induction, $\bigcap_{k=1}^{n} I_k = \prod_{k=1}^{n} I_k$ is proved.

Theorem 3.12 *If $P_1, \cdots, P_n$ are comaximal prime ideals over a semiring S then $I = P_1 \ldots P_n$ is n-absorbing. Moreover, $\omega(I) = n$.*

Proof. Note that $I = P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$ by the above remark. and By

Theorem 1.9 we have that I is n-absorbing. The moreover statement follows from remark theorem 3.7 since they are incomparable.

In general the product of n prime ideals is not n-absorbing. Badawi provides some examples which we mention some of here:

Example 3.13 *let $R = \mathbb{Z}[X,Y,Z]$. Let $P_1 = (2,X)$, $P_2 = (2,Y)$ and $P_3(2,Z)$. These are incomparable prime ideals (but non maximal). The product $I = P_1P_2P_3 = (2,4X,4Y,4Z,2XY,2XZ,2YZ,XYZ)$ is not 3-absorbing. To show that take $x_1 = 2$, $x_2 = X+Y+2$, $x_3 = X+Z+2$ and $x_4 = Y+Z+2$. then $x_1x_2x_3x_4 = 16+16X+4X^2+16Y+12XY+2X^2Y+4Y^2+2XY^2+16Z+12XZ+2X^2Z+12YZ+4XYZ+2Y^2Z+4Z^2+2XZ^2+2YZ^2$. This is clearly in I since $2 \in I$ and the above can me written as $2x$ where $x \in \mathbb{Z}[X,Y,Z]$. We claim that no other product of 3 elements is in Z.*

To show this let's examine the possible 3 products which are $x_1x_2x_4$, $x_1x_2x_3$, $x_1x_3x_4$ and $x_3x_2x_4$. We list the polynomials here without a specific order and show that each of them is not in I.

1. $8+8X+2X^2+4Y+2XY+4Z+2XZ+2YZ$:

   suppose that $8+8X+2X^2+4Y+2XY+4Z+2XZ+2YZ \in I$. this would imply that $2X^2 \in I$ which is not the case.

2. $2XY+2XZ+4X+2Y^2+2YZ+8Y+4Z+8$:

   if this is in I this would make a contradiction, since $2Y^2 \notin I$.

3. $2XY+2XZ+4X+2YZ+4Y+2Z^2+8Z+8$:

   same as above.

4. $8+8X+2X^2+8Y+6XY+X^2Y+2Y^2+XY^2+8Z+6XZ+X^2Z+6YZ+2XYZ+X^2Z+2Z^2+XZ^2+YZ^2$:

   if this is in I *this would imply that* $2X^2+2Y^2+XY^2+X^2Z+2Z^2+XZ^2+YZ^2 \in I$ *which again isn't in I. so* $YZ^2+XY^2+XZ^2+X^2Z \in I$ *Which is a contradiction since this can't be factorized as a sum of elements in* $\{2,4X,4Y,4Z,2XY,2XZ,2YZ,XYZ\}$.

We also want to give an example where $\omega(IJ) < \omega(I)+\omega(J)$. This can be shown with an example:

Example 3.14 *Let* I *be an Idempotent prime ideal and let* $J = I$*. In this case,* $\omega(IJ) = \omega(I^2) = \omega(I) < \omega(I)+\omega(J)$.

Proposition 3.15 ([9]) *Let* S *be a semiring and* I, J *be ideals of* S*. Then the following statements hold:*

1. $Rad(I) \subseteq Rad(I)$ *and* $Rad(I) = Rad(Rad(I))$.
2. $Rad(IJ) = Rad(I \cap J) = Rad(I) \cap Rad(J)$.
3. $Rad(I) = S$ *iff* $I = S$.
4. $Rad(I+J) = Rad(I)+Rad(J)$.

Lemma 3.16 *Suppose* I, J *are two distinct maximal ideals in a semiring* S*, Then* $I^n + J^m = S$.

Proof. Given I, J ideals of S such that $I+J = S$ then $Rad(J^m+I^n) = Rad(Rad(J^m)+Rad(I^n)) = Rad(I+J) = Rad(S) = S$. But $Rad(J^m+I^n) = S$ iff $J^m+I^n = S$.

Lemma 3.17 *Let M be a maximal ideal in a semiring S. Then for any element $x \notin M$, $(M^n, x) = S$.*

Proof. Note that $Rad(M^n + (x)) = Rad(Rad(M^n) + Rad(x)) = Rad(M) + Rad((x)) = M + Rad((x))$ and since $M \subset M + Rad((x))$ we conclude that $Rad(M^n + (x)) = S$.

Lemma 3.18 *Let M be a maximal ideal of a semiring S and n a positive integer. Then M^n is an n-absorbing ideal of S. Moreover, $\omega(I) \leq n$ and $\omega(I) = n$ if $M^{n+1} \subset M^n$.*

proof. Let $x_1 \cdots x_{n+1} \in M^n$. If $x_1, \cdots, x_{n+1} \in M$ then we are done. So suppose at least one element, say, $x_{n+1} \notin M$. By the lemma above we have $(M^n, x_{n+1}) = S$. that is, $m + sx_{n+1} = 1$ where $m \in M$ and $s \in S$. So $x_1 \cdots x_n = (x_1 \cdots x_n)1 = (x_1 \cdots x_n)(m + sx_{n+1}) = (x_1 \ldots x_n)m + (x_1 \cdots x_{n+1}) \in M^n$. So M^n is n-absorbing. The first part of the moreover statement is trivial. To show the second part is true We want to show that M^n is not $(n-1)$-absorbing. suppose that $M^{n+1} \subset M^n$ and that M^n is $(n-1)$-absorbing. Then it's possible to choose $x_1, \cdots, x_n \in M$ such that $x_1 \cdots x_n \in M^n \backslash M^{n+1}$. M^n is n-absorbing so that we have a product of $n-1$ of the x_i's M^n. But this would imply that $x_1 \cdots x_n \in M^{n+1}$ which is a contradiction to our previous assumption. So combining this with the fact that M^n is n-absorbing we conclude that $\omega(M^n) = n$.

Theorem 3.19 *Let $M_1, \cdots M_n$ be maximal ideals of a semiring S then $I = M_1 \cdots M_n$ is an n-absorbing ideal of S. Moreover $\omega(I) \leq n$.*

Proof. We can show the following holds. Let $M_1, \cdots, M_m$ be distinct maximal ideals of a semiring S. Let $n_1 + \cdots + n_m = n$. Then defining $I = M_1^{n_1} \cdots M_m^{n_m}$ we

note that $M_i^{n_i}$ is n_i absorbing (Previous lemma). so $I = M_1^{n_1} \cdots M_m^{n_m} = M_1^{n_1} \cdots M_m^{n_m}$. and by theorem 1.18(c) It follows that I is n-absorbing.
The moreover statement is clear.

Lemma 3.20 *Let $P_1, \cdots, P_n$ be incomparable prime ideals in a semiring S. Let I be an n-absorbing ideal contained in the intersection of these ideals. If $x_1^{m_1} \cdots x_n^{m_n} \in I$ where m_i's are positive integers and $x_i \in P_i - \bigcup_{k \neq i} P_k$ then $x_1 \cdots x_n \in I$.*

Proof: Since I is assumed to be n-absorbing we must have $x_1^{k_1} \cdots x_n^{k_n} \in I$ With $k_1 + \cdots + k_n = n$. If any $k_i = 0$, say k_1 we have that $x_2^{k_2} \cdots x_n^{k_n} \in I \subseteq P_1$. And since P_1 was chosen prime, this implies $x_i \in P_1, i \neq 1$, a contradiction.

Lemma 3.21 *Let I be an n-absorbing k-ideal with $n \geq 2$. Suppose I has exactly n minimal prime k-ideals over it, say $P_1, \cdots, P_n$. Let $c_i \in P_i - \bigcup_{j \neq i} P_j$ for $1 \leq i \leq n$ then $P_j \prod_{i \neq j} c_i \subseteq I$.*

Proof. First suppose $a_j \in P_j - \bigcup_{i \neq j} P_i$, Then since $Rad(I)$ is in the intersection of all prime ideals over I and using Theorem 1 e and the previous lemma we have $a_j \prod_{i \neq j} c_i \in I$. Now suppose that $a \in P_j \cap (\bigcup_{i \neq j} P_i)$. Now let $d \in (P_j - \bigcup_{i \neq j} P_i)$. We wish to choose an element b such that $a + bd \in (P_j - \bigcup_{i \neq j} P_i)$. Let $F = \{m | a \in P_i$ for $1 \leq m \leq n\}$ and let $D = \{m | a \notin P_i$ for $1 \leq m \leq n, m \neq j\}$. If $F = \emptyset$ choose b =1. Otherwise, let $b = \prod_{k \in F} c_k$. Now since $bd \in P_m$ for all $m \in F$ and $a \notin P_m$ for every $m \in F$, We conclude $(bd + a) \notin P_m$ for every $m \in F$(Since Every P_m is a k-ideal). And since $a \in P_m$ for every $m \in D$ and $bd \notin P_m$ for all $m \in D$, We get $(bd + a) \notin P_m$ for all $m \in D$(Since P_m is a k-deal). And since our choice of d,a ensures they are in P_j we conclude that $bd + a \in P_j - (\bigcup_{i \neq j} P_i)$. Let $x = bd + a$. As in the first part of this proof $x\prod_{i\neq j} c_i \in I$ and $d\prod_{i\neq j} c_i \in I$. Now since $(\prod_{k\in F} c_k)(d\prod_{i\neq j} c_i) + a\prod_{i\neq j} c_i = x\prod_{i\neq j} c_i \in I$. We can conclude, since I is a k-ideal, that $a\prod_{i\neq j} c_i \in I$.

Corollary 3.22 *For $n \geq 2$, Let $P_1, \cdots, P_n$ be incomparable prime k-ideals of a semiring S. Let $a \in P_j$ for some $1 \leq j \leq n$. Then there is an element $d \in P_j - \bigcup_{i\neq j} P_i$ and $b \in S$ such that $(bd + a) \in P_j - \bigcup_{i\neq j} P_i$.*

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Theorem 3.23 *Let I be an n-absorbing k-ideal of a semiring S such that I has n minimal prime k-ideals, say $P_1, \cdots, P_n$. Then $P_1 \cdots P_n \subseteq I$. Moreover $\omega(I) = n$.*

Proof. Choose $a_i \in P_i$ for $1 \leq i \leq n$. Now for any choice of $c_i \in P_i \backslash (P_1 \cup \bigcup_{j\neq i} P_j)$ we get $a_i \prod_{2\leq i\leq n} c_i \in I$ by the previous lemma. Now we want to use induction. Suppose that for some k, $1 \leq k \leq n-1$ we have $(a_1 \cdots a_k) \prod_{k+1\leq i\leq n} c_i \in I$ for any choice of $c_i \in P_i \backslash (P_1 \cup \bigcup_{j\neq i} P_j)$ Where $k+1 \leq i \leq n$. We will show that $(a_1 \cdots a_{k+1}) \prod_{k+2\leq i\leq n} c_i \in I$ for any choice of $c_i \in P_i \backslash (P_1 \cup \bigcup_{j\neq i} P_j)$. where $k+2 \leq i \leq n$. By the previous corollary we may $d_{k+1} \in P_{k+1} \backslash \cup_{j\neq k+1} P_j$ and $b_{k+1} \in S$ such that $b_{k+1}d_{k+1} + a_{k+1} \in P_{k+1} \backslash \cup_{j\neq k+1} P_j$. Choose $c_{k+1} = b_{k+1}d_{k+1} + a_{k+1}$. By the assumption above we have $(a_1 \cdots a_k) \prod_{k+1\leq i\leq n} c_i = (a_1 \cdots a_k)c_{k+1} \prod_{k+2\leq i\leq n} c_i = (b_{k+1}a_1 \cdots a_k d_{k+1}) \prod_{k+2\leq i\leq n} c_i + (a_1 \cdots a_k a_{k+1}) \prod_{k+2\leq i\leq n} c_i) \in I$.

Now $b_k(a_1 \cdots a_k)d_{k+1} \prod_{(k+2)\leq i\leq n} c_i \in I$ by assumption (by the choice of d_{k+1}). and since I is a k-ideal it follows that $(a_1 \cdots a_{k+1}) \prod_{(k+2)\leq i\leq n} c_i \in I$. To prove the statement of the theorem, set $k = n-1$ and $c_n = b_n d_n + a_n$ then $(a_1 \cdots a_{n-1}(c_n) \in I$ again applying the same argument we get $(a_1 \cdots a_n) \in I$. that is $P_1 \cdots P_n \subseteq I$, Since each a_i was chosen arbitrarily from P_i.

To show $\omega(I) = n$, choose $x_i \in P_i \setminus \bigcap_{j\neq i} P_j$. Now $x_i \cdots x_n \in I$ by above ($P_1 \ldots P_n \subseteq I$). Suppose a product of $n-1$ of the x_i's is in I, say, $x_2 \cdots x_n \in I \subseteq P_1$, then this would imply $x_i \in P_1$ where $2 \leq n$ which contradicts the choice of x_i's. So we have $\omega(I) = n$.

Corollary : Let I be an n-absorbing k-ideal of a semiring S such that I has exactly n minimal prime ideals, say $P_1, \cdots P_n$. If the P_i's are comaximal, then $I = P_1 \cdots P_n$.

Proof. This follows from the fact that $P_1 \cdots P_n \subseteq I \subseteq P_1 \cap \cdots \cap P_n$ from previous theorem. And since $P_1, \cdots P_n$ are comaximal we have $P_1 \cap \cdots \cap P_n = P_1 \cdots P_n$ (Remark 4.6). We now have $P_1 \cdots P_n \subseteq I \subseteq P_1 \cdots P_n$ and thus $I = P_1 \cdots P_n$.

Theorem 3.24 *Suppose that P is a prime ideal of a semiring S. Let I be a P-primary ideal of S such that $P^n \subseteq I$ for some natural number n. Then I is an n-absorbing ideal of S. Moreover, $\omega(I) \leq n$ In particular if P^n is P-primary ideal of S then it is n-absorbing with $\omega(P^n) \leq n$. If $P^{n+1} \subset P^n$ then $\omega(P^n) = n$*

Proof. Let $x_1 \cdots x_{n+1} \in I$. If any of the x_i's is not in P then, since I is $P-primary$, we conclude that the product of the others is in I and we are done. If all of the x_i's are in P then by the assumption that $P^n \subseteq I$ we have $x_1 \cdots x_n \in I$ (any product of n of x_i is in I). The proof of the moreover statement follows as in lemma 4.12.

An example of a prime ideal P such that P^2 is 2-absorbing but P^2 is not primary is given in [3]. We list it here but before we show a theorem from the same paper without it's proof:

Theorem 3.25 *Suppose that I is an ideal of R such that $I \neq Rad(I)$ and $Rad(I)$ is a prime ideal of R. Then the following statements are equivalent:*

1. I is a 2-absorbing ideal of R.

2. $B_x = \{x \in R : yx \in I\}$ is a prime ideal of R for each $x \in Rad(I)\backslash I$.

Example 3.26 *Let $R = \mathbb{Z}+3X\mathbb{Z}[X]$. and let $P = 3X\mathbb{Z}[X]$. Then $P^2 = 9X^2\mathbb{Z}[X]$. Note that $3(3x^2) \in P^2$. and since $3 \notin P^2$ and $3x^2 \notin P^2$ and neither is $(3x^2)^n$ or 3^n for any $n \geq 1$, we can say that P^2 is not a primary ideal.*

Theorem 3.27 *Let P be a divided prime ideal of a semiring S. And let I be an n-absorbing ideal of S such that $Rad(I) = P$. Then I is P-primary.*

Proof. To show this is primary Let $xy \in I$ with $x, y \in S$ and $y \notin P$. Now $P \subset y^{n-1}S$ since P is a divided ideal in S and $y^{n-1} \notin P$ we have that $x = y^{n-1}z$. Note that $zy^n = xy \in I$. And since I is n-absorbing and $y^n \notin I$ (Otherwise $y \in P$ by definition of the radical). We conclude that $xy^{n-1} \in I$. Hence I is a P-primary ideal of S.

For an ideal I of a semiring S define $I_x = \{y \in S \,|\, xy \in I\} = (I :_x S)$. This is usually called an ideal quotient.

Theorem 3.28 *Let I be an n-absorbing ideal of a semiring S. Then $I_y = (I :_S y)$ is also an n-absorbing ideal of the semiring S for all $y \notin I$.*

Proof : Suppose that I is an n-absorbing ideal S. and let $x_1 \cdots x_{n+1} \in I_y$ for $x_1, \cdots, x_{n+1}$. Then, $yx_1 \cdots x_{n+1} = (yx_1)x_2 \cdots x_{n+1} \in I$. If $x_2 \cdots x_n + 1$ we are done. So assume not. We must have y times $n-1$ of the $x_i's$, $2 \leq i \leq n$. say, without loss of generality, $yx_1 \cdots x_{n-1}$. and so by definition of I_x, $x_1 \cdots\cdots x_{n-1} \in I_x$ as desired. So I_x is an n-absorbing ideal of S. The moreover statement is clear.

Theorem 3.29 *Let $n \geq 2$ and $I \subset Rad(S)$ be an n-absorbing k-ideal of a semiring S. Suppose that $x \in Rad(I) \backslash I$, and let $m \geq 2$ be the least positive integer such that $x^m \in I$. Then $I_{x^{m-1}} = (I :_S x^{m-1})$ is an $n - m + 1$ absorbing ideal of S containing I.*

Proof. Suppose that $x_1 \cdots x_{n-m+2} \in I_{x^{m-1}}$ for $x_1, \cdots, x_{n-m+2} \in S$. By definition of $I_{x^{m-1}}$ we have $x^{m-1}x_1 \cdots x_{n-m+1} \in I$. Since I is n-absorbing, either $x^{m-2}x_1 \cdots x_{n-m+1} \in I$ or x^{m-2} multiplied by a combination of $n-m+2$ elements of the x_i's. If the later holds then we are done. So assume that $x^{m-2}x_1 \cdots x_{n-m+1} \in I$. Since we assumed $x^m \in I$. We now have $x(x^{m-2}x_1 \cdots x_{n-m+2}) \in I$ and $x^m(x_1 \cdots x_{n-m+2}) \in I$ and so we have the $xx^{m-2}x_1 \cdots x_{n-m+1}(x_{n-m+2}+x) \in I$. again, using the fact that I is n-absorbing and the fact that a product of x^{m-1} and n-m+1 of the elements is not in I, we have $x^{m-2}x_1 \cdots x_{n-m+1}(x_{n-m+2} + x) \in I$. Since I is a k-ideal and $x^{m-2}x_1 \cdots x_{n-m+2} \in I$ we conclude $x^{m-1}x_1 \cdots x_{n-m+1}$ which is a contradiction as we assumed no product of $n - m + 1$ of the x_i's.

Corollary 3.30 *If I is an n-absorbing ideal of a semiring S. Let $x \in Rad(I) \backslash I$ and suppose that n is the least element such that $x^n \in I$ then $I_{x^n} = (I :_S x^{n-1})$ is a prime ideal in I.*

Proof. By the above theorem we know that I_{x^n} is $(n - n + 1)$-absorbing i.e it is 1-absorbing and thus a prime ideal containing Rad(I).

Corollary 3.31 *Let $n \geq 2$ and I be an n-absorbing P-primary ideal of a semiring S for some prime ideal P of S. If $x \in Rad(I) \backslash I$ and n is the least positive integer such that $x^n \in I$, then $I_{x^{n-1}} = (I :_S x^{n-1}) = P$.*

proof. By the previous corollary, we have $P = Rad(I) \subseteq I_{x^{n-1}}$. To show that $I_{x^{n-1}} \subseteq P$ choose $y \in I_{x^{n-1}}$. so $yx^{n-1} \in I$. since $x^{n-1} \notin I$ then, since I is P-primary, $y \in P$. So this concludes $I_{x^{n-1}} \in P$.

Theorem 3.32 *Let $n \geq 2$ and $I \subset Rad(I)$ be be an n-absorbing ideal of a semiring S such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$ Suppose that $x \in Rad(I) \backslash I$. And let $m \geq 2$ be the least positive integer such that $x^m \in I$ Then every product of $n - m + 1$ of the ideals $P_1, \cdots, P_n$ is contained in $I_{x^m - 1} = (I :_S x^{m-1})$.*

Proof. First we show that $m \leq n$. Suppose not, let $m > n$ be the least positive integer satisfying $x^m \in I$. then since I is n-absorbing we must conclude that $x^n \in I$ which contradicts the minimality of m. So we must have $n - m + 1 \geq 1$. Let $F = \{Q_1, \cdots, Q_{m-1}\} \subset G = \{P_1, \cdots, P_n\}$ and let $D = G \backslash F$. This way D contains exactly $n - m + 1$ copies of the $P_i' s$. If $x \in Rad(I) \backslash I$ then $x \in Q$ for all $Q \in F$. This holds since the radical is the intersection of prime ideals over I. From this we have $x^{m-1} \in \prod_{i=1}^{m-1} Q_i$. By previous theorem, we have $\prod_{Q_i \in F} Q_i \prod_{P_i \in G} P_i = P_1 \cdots P_n \subseteq I$. So $x^{m-1} \prod_{P_i \in F} P_i \subseteq I$. So we conclude $\prod_{P_i \in F} P_i \subseteq I_{x^{m-1}}$ as desired.

Theorem 3.33 *Let $n \geq 2$ and $I \subset Rad(I)$ be be an n-absorbing ideal of a semiring S such that I has exactly n minimal prime ideals, say $P_1, \ldots, P_n$ Suppose that $x \in Rad(I) \backslash I$. Then every product of $n - 1$ of the ideals $P_1, \cdots, P_n$ is contained in $I_x = (I :_S x)$.*

Proof. From the above theorem, we either have $x^2 \in I$ with 2 being the least positive number m such that $x^m \in I$, in which case we have that every product of $n - 2 + 1 = n - 1$ of the ideals $P_1, \cdots, P_n$ is contained in $I_x = (I :_S x)$. Otherwise, we can do a similar proof of the above theorem.

In this section we explore which of the results in Badawi concerning the ring theoretic constructions. And the first we might ask is whether the preimage of an ideal of a semiring is also an ideal. To state the question with more formality: Let $f : S \to T$ be a semiring homomorphism and let J be an ideal of T. Is $f^{-1}(J)$ an ideal. The answer is yes. To show so suppose that $x, y \in f^{-1}(J)$. Then $f(x), f(y) \in J$ and so $f(x) + f(y) = f(x + y) \in J$ and thus $x + y \in f^{-1}(J)$. Based on this we have the following result:

Theorem 3.34 *Let S and T be semirings and let J be an n-absorbing ideal of T. Let $f : S \to T$ be a semiring homomorphism then $I = f^{-1}(J)$ is an n-absorbing ideal of S. Moreover, $\omega_R(f^{-1}(J)) \leq \omega_T(J)$.*

Proof. Let $x_1 \cdots x_{n+1} \in I$ where $x_1, \cdots, x_{n+1} \in S$. then $f(x_1 \cdots x_{n+1}) \in J$. And so $f(x_1) \cdots f(x_{n+1}) \in J$. And since J is n-absorbing we conclude the product of n of them, say without loss of generality $f(x_1) \cdots f(x_n) \in J$ and so $f(x_1 \cdots x_n) \in J$ thus $x_1 \cdots x_n \in f^{-1}(J)$.
The moreover statement is clear.

Theorem 3.35 *Let S_1 and S_2 be semirings. And let I_1 be an n-absorbing ideal of S_1 and I_2 be an m-absorbing ideal of S_2. Then $I_1 \times I_2$ is an $m + n$ absorbing ideal of $S_1 \times S_2$. Moreover, $\omega_{S_1 \times S_2}(I_1 \times I_2) = \omega_{S_1}(I_1) + \omega_{S_2}(I_2)$.*

Proof. Suppose that $z_1 \cdots z_{m+n+1} \in I_1 \times I_2$ and note that each $z_i = (x_i, y_i)$ where $x_i \in I_1$ and $y_i \in I_2$. Now $z_1 \cdots z_{n+m+1} = (x_1 \cdots x_{n+m+1}, y_1 \cdots y_{m+n+1} \in I_1 \times I_2)$. Hence $x_1 \cdots x_{n+m+1} \in I_1$ and $y_1 \cdots y_{n+m+1} \in I_2$. but since I_1 and I_2 are n-absorbing and m-absorbing respectively. We conclude that n of the $x_i's$ have their product in I_1 (say $x_{i_1} \cdots x_{i_n}$) and m of the y_i's have their product in I_2 (say $y_{k_1} \cdots y_{k_m}$). Now assuming K = $\{i_1, \cdots .i_n\} \cup \{k_1 \cdots k_m\}$

then $\prod_{k \in K}(x_k, y_k) \in I_1 \times I_2$ (the proof is valid even though K might have less than n+m elements by theorem 1).

We now show the moreover statement is correct:

Corollary 3.36 *Let $S_1, \cdots, S_n$ be semirings and let I_k be an ideal of S_k for $1 \leq k \leq n$. Then If I_k is m_k-absorbing then $I_1 \times \cdots \times I_n$ is ($m_1 + \cdots + m_n$)-absorbing.*

Proof. Suppose that this holds for $n = l$. we show it holds for $n = l + 1$. If $S_1, \cdots S_{l+1}$ are semirings with corresponding Ideals $I_1, \cdots, I_{l+1}$. Note that $I_1 \times \cdots \times I_l$ is an $(n_1 + \cdots + n_l)$-absorbing ideal of $S_1 \times \cdots \times S_l$ by assumption and by previous theorem we have $(I_1 \times \cdots \times I_l) \times I_{l+1}$ to be an $(n_1 + \cdots + n_{l+1})$-absorbing ideal of the semiring $S_1 \times \cdots \times S_{l+1}$.

4 Conclusion

In this thesis we recalled some of the algebraic structures in semiring theory and gave some examples related to it. We studied the concept of n-absorbing ideals in commutative semirings and illustrated it with many examples and introduced some propositions. As well as generalizing some of the results that don't hold for normal ideals to subtractive ideals. Most of the work presented in this thesis is a generalization to the work done by Badawi and Anderson.

5 Future Work

In the future, we wish to study the results to some specific types of semirings as well as prufer domains. It is possible to also consider to generalize some results that require some workarounds.

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